Tractable and asymptotic behaviour of mixed Wiener spaces

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Joint work with...

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Diese Maßnahme wird mitfinanziert durch Steuermittel auf der Grundlage des vom Sächsischen Landtag beschlossenen Haushaltes.

Asymptotic characteristics

For $n \in \mathbb{N}_0$, and $X(\Omega), Y$ quasi-Banach function spaces with a continuous linear embedding $T : X \to Y$ the following (quasi) s-Numbers are defined:

- Sampling numbers (linear and non-linear)

$$\varrho_n(X)_Y = \inf_{t_1 \dots t_n \in \Omega} \inf_{R:\mathbb{C}^n \to Y} \sup_{\|f\|_X \leq 1} \big\| f - R\big(f(t_1) \dots f(t_n)\big) \big\|_Y \tag{1}$$

- Gelfand numbers

$$c_n(X)_Y = \inf \left\{ \sup_{f \in B_X \cap M} \|f\|_Y : M \subset X \text{ linear subspace with } \operatorname{codim} M < n \right\} \tag{2}$$

- best trigonometric m-term approximation

$$\sigma_n(X)_Y := \sup_{\|f\|_X \leq 1} \inf_{s \in \Sigma_n} \|f - s\|_Y \tag{3}$$

Main result - Tractable s-numbers

The (first) main contribution of this talk are the following tractable results

$$\sigma_n\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_\infty \leq Cn^{-\frac{1}{2}} d \log(n) \tag{4}$$

and

$$c_n\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_2 \leq Cn^{-\frac{1}{2}} d \log(n) \tag{5}$$

with absolute constants C. Note that the right hand side only depends linearly on the dimension d of the space.

Motivation

- Mixed Wiener spaces are easier to study than dominating mixed smoothness Besov spaces and can be embedded into these
- Wiener classes (isotropic) and their embeddings have been studied by Temlayakov, Krieg and others
- Nguyen Nguyen and Sickel studied s-numbers of mixed Wiener classes in [7], however they studied neither Gelfand numbers, sampling numbers nor best m-term approximation
- new results concerning sampling numbers

Proposition 1 ([4, Jahn, T. Ullrich and Voigtlaender 2023])

Let $n, d \in \mathbb{N}$ *then it holds for a quasi-normed function space with continuous embedding into* L_∞

$$\varrho_{nd \log(d) \log(n)^2 \log(N)}\big(\mathcal{F}\big)_2 \leq C\Big(\sigma_n\big(\mathcal{F}\big)_\infty + E_{[-N,N]^d}(\mathcal{F})_\infty\Big). \tag{6}$$

See also [5] by Krieg for a version with $\mathcal{A}$ as target space om the right hand side.

Relations between s-numbers

- Gelfand numbers form a lower bound for the non-linear sampling numbers, in particular it holds
$$\varrho_n\big(\mathcal{A}_p^\alpha(\mathbb{T}^d)\big)_2 \gtrsim c_n(\mathcal{A}_p^\alpha(\mathbb{T}^d))_2$$
- Kolmogorov numbers form a lower bound for the linear sampling numbers, in particular it holds
$$\varrho_n^{\mathrm{lin}}\big(\mathcal{A}_p^\alpha(\mathbb{T}^d)\big)_2 \gtrsim d_n(\mathcal{A}_p^\alpha(\mathbb{T}^d))_2$$

In total the Gelfand and sampling numbers give upper and lower bounds for the non-linear sampling numbers.

Mixed Wiener spaces

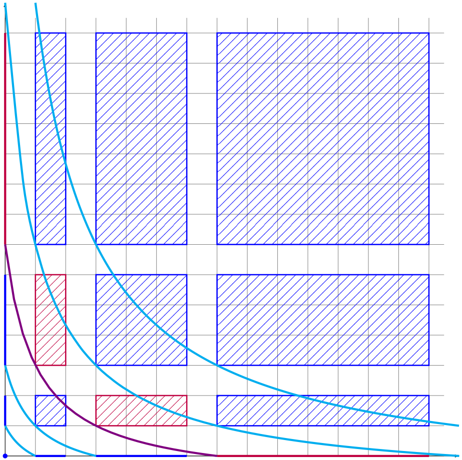
For $\alpha > 0$ and $0 < p < \infty$ we define the mixed Wiener space $\mathcal{A}_p^\alpha(\mathbb{T}^d) \subset L_1(\mathbb{T}^d)$ via its norm

$$\|f\|_{\mathcal{A}_p^\alpha(\mathbb{T}^d)} = \Big(\sum_{\mathbf{k}\in\mathbb{Z}^d} \prod_{i=1}^{d} (1 + |k_i|)^{\alpha p} |\hat{f}(\mathbf{k})|^p \Big)^{\frac{1}{p}}.$$

For $p = 1$ these spaces are the periodic versions of Barron classes. The space $\mathcal{A}_1^0$ is the original Wiener Algebra $\mathcal{A}$. They have a useful embedding into the sequence spaces

$$A_\alpha f = \Big(\prod_{i=1}^{d} (1 + |k_i|)^\alpha \hat{f}(\mathbf{k}) \Big)_{\mathbf{k}\in\mathbb{Z}^d}, \quad \|A_\alpha : \mathcal{A}_p^\alpha(\mathbb{T}^d) \to \ell_p(\mathbb{Z}^d)\| = 1.$$

Hyperbolic cross



A hyperbolic cross is a set of the form $\left\{\mathbf{n} \in \mathbb{N}_0^d \middle| \prod_{j=1}^d (n_j + 1) \leq c\right\}$.
These are exactly the balls of this norm on the frequency side (for functions with only one Fourier coefficient).
For our proofs we are particularly interested in dyadic hyperbolic crosses.

Theorem 2

For $n, d \in \mathbb{N}$, $0 < p \leq 2$ *and* $\alpha > \left(\frac{p-1}{p}\right)_+$ *it holds*

$$c_n(\mathcal{A}_p^\alpha(\mathbb{T}^d))_2 \asymp n^{-(\alpha+\frac{1}{p}-\frac{1}{2})} \log(n)^{(d-1)\alpha}. \tag{7}$$

Theorem 3

For $n, d \in \mathbb{N}$ *with* $0 < p \leq q$ *and* $2 \leq q \leq \infty$ *as well as* $\alpha > \left(\frac{p-1}{p}\right)_+$ *it holds*

$$n^{-(\alpha+\frac{1}{p}-\frac{1}{2})} \log(n)^{(d-1)\alpha} \lesssim \sigma_n\big(\mathcal{A}_p^\alpha(\mathbb{T}^d)\big)_q \lesssim n^{-(\alpha+\frac{1}{p}-\frac{1}{2})} \log(n)^{(d-1)\alpha+\mu} \tag{8}$$

where $\mu = \frac{1}{2}$ *if both* $q = \infty$ *and* $d > 1$ *otherwise* $\mu = 0$.

Linear sampling numbers

Proposition 4 (see [7, Nguyen, Nguyen and Sickel, 2022])

For the Kolmogorov numbers d_n it holds for $\alpha > 0$,

$$d_n\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_2 \asymp n^{-\alpha}\log(n)^{\alpha(d-1)}. \tag{9}$$

Since the Kolmogorov numbers form a lower bound for the linear sampling numbers this immediately gives the following result

$$\varrho_n^{\mathrm{lin}}\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_2 \gtrsim n^{-\alpha}\log(n)^{\alpha(d-1)}. \tag{10}$$

Non-linear sampling numbers

For the non-linear sampling numbers an analogous bound holds in terms of the Gelfand numbers

$$\varrho_n\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_2 \gtrsim n^{-(\alpha+\frac{1}{2})}\log(n)^{\alpha(d-1)}. \tag{11}$$

Proposition 1 together with Theorem 3 now yields

$$\varrho_n\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_2 \lesssim n^{-(\alpha+\frac{1}{2})}\log(n)^{\alpha(d-1)+3(\alpha+\frac{1}{2})+\frac{1}{2}} \tag{12}$$

There is a difference of $\frac{1}{2}$ in the main rate of the decay between the linear and non-linear sampling numbers in mixed Wiener classes measured in L_2.

Best m-term approximation of $\mathcal{A}$

Lemma 5

Let $2 \leq q < \infty$ *and* $\alpha > 0$ *then it holds*

$$\sigma_n(\mathcal{A})_q \leq C\sqrt{\frac{q}{n}} \tag{13}$$

for an absolute constant $C \leq 47$.

We can even employ the Nikolskij inequality to get a version of this for $q = \infty$.

Lemma 6

For $N \in \mathbb{N}$ *and a trigonometric polynomial* $t \in \mathcal{T}([-N, N]^d)$ *it holds*

$$\sigma_n(t)_\infty \leq Cd\log(N)n^{-\frac{1}{2}}\|t\|_{\mathcal{A}}. \tag{14}$$

Tractable bound on the best m-term approximation

Again the original Theorem 3 states

$$\sigma_n\left(\mathcal{A}_1^\alpha(\mathbb{T}^d)\right)_\infty \lesssim n^{-(\alpha+\frac{1}{2})}\log(n)^{(d-1)\alpha+\frac{1}{2}}$$

Where another 2^d term is hidden by the $\lesssim$. This is not a suitable bound in a setting where $n = d^s$.

Theorem 7

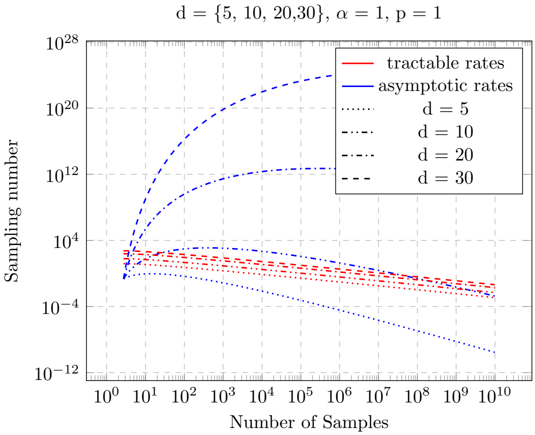
Let $m, d \in \mathbb{N}$ and $\alpha > 0$ then it holds

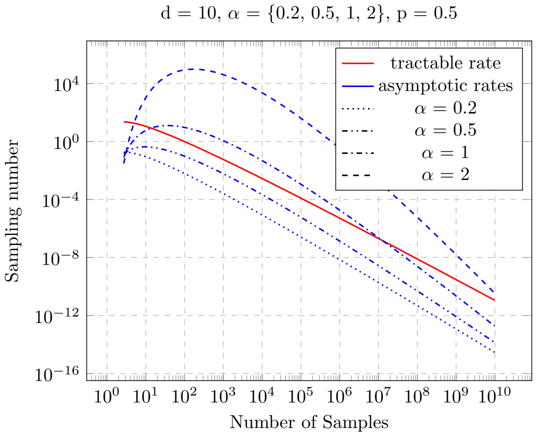
$$\sigma_n\left(\mathcal{A}_1^\alpha(\mathbb{T}^d)\right)_\infty \leq Cn^{-\frac{1}{2}}d\log(n)^{1/2} \tag{15}$$

with absolute constant $C \geq 1$.

By using [2]. This bound decays for $n > d^2$. An asymptotic version of this can be found in [9] without comments on the d-dependence.

Comparison of rates





Thank you for your attention

References

- Feng Dai, Vladimir N. Temylakov
  Random points are good for universal discretization
  J. Math. Anal. Appl, 529(1), 2024
- Ronald A. DeVore and Vladimir N. Temlyakov
  Nonlinear Approximation by Trigonometric Sums
  Journal of Fourier Analysis and Applications 1995
- Simon Foucart, Alain Pajor, Holger Rauhut, Tino Ullrich
  The Gelfand widths of ℓ_p-balls for $0 < p \leq 1$
  Journal of Complexity, Volume 26, Issue 6, December 2010, Pages 629-640
- Thomas Jahn, Tino Ullrich, Felix Voigtlaender
  Sampling numbers of smoothness classes via ℓ_1-minimization
  Journal of Complexity, arxiv.2212.00445 2023
- David Krieg
  Tractability of sampling recovery on unweighted function classes
  Arxiv 2023
- M. M., Serhii Stasyuk, Tino Ullrich
  Gelfand numbers and best m-term trigonometric approximation of weighted Wiener classes
  work in progress
- Van Dung Nguyen, Van Kien Nguyen, and Winfried Sickel
  s-Numbers of Embeddings of Weighted Wiener Algebras
  Journal of Approximation Theory, Volume 279, July 2022, 105745
- Vladimir N. Temlyakov
  Approximations of functions with bounded mixed derivative
  Trudy Mat. Inst. Steklov. volume 178, page 92, 1986
- Vladimir N. Temlyakov
  Constructive sparse trigonometric approximation and other problems for functions with mixed smoothness
  Sbornik: Mathematics 2015

Besov spaces

For Besov spaces with dominating mixed smoothness

$$S^r_{p,\theta}B(\mathbb{T}^d) := \Big\{ f \in L_p(\mathbb{T}^d) \ : \Big(\sum_{\mathbf{l}\in\mathbb{N}_0^d} 2^{|\mathbf{l}|_1 r\theta} \Big\| \sum_{\mathbf{k}\in I_{\mathbf{l}}} \hat{f}(\mathbf{k}) \exp(2\pi i \mathbf{k}\mathbf{x}) \Big\|_p^\theta \Big)^{\frac{1}{\theta}} < \infty \Big\}, \tag{16}$$

we get with $I_0 = \{0\}^d$ and for $\mathbf{l} \in \mathbb{N}^d$,

$$I_{\mathbf{l}} = \big\{ k \in \mathbb{Z} \ : \ 2^{l_1 - 1} \le |k| < 2^{l_1} \big\} \times \cdots \times \big\{ k \in \mathbb{Z} \ : \ 2^{l_d - 1} \le |k| < 2^{l_d} \big\} .$$

Theorem 8

Let $n, d \in \mathbb{N}$ *and* $(p, \theta) \in \{(p,\theta) : \ 2 \le p < \infty, 0 < \theta \le 1\} \setminus (2, 1)$. *Then it holds*

$$\sigma_n(S^{1/\theta-1/2}_{p,\theta}B(\mathbb{T}^d))_\infty \le C \, (1/\theta - 1/2 - 1/p)^{-1/2} d \, n^{1/2-1/\theta} \log(dn)^{1/2} \, , \tag{17}$$

where $C > 0$ *denotes an absolute constant.*

Besov spaces embedded into Wiener spaces

Idea of proof: The space $S^{1/\theta-1/2}_{p,\theta}B(\mathbb{T}^d)$ is embedded into $S^0_\theta\mathcal{A}(\mathbb{T}^d)$ for $2 \leq p < \infty$ and $0 < \theta \leq 1$ with constant operator norm.

$$\begin{array}{ccc} S^{1/\theta-1/2}_{p,\theta}B & \xrightarrow{\ ?\ } & L_\infty \\ {\scriptstyle 1}\downarrow & & \uparrow{\scriptstyle m^{-1/2}} \\ S^0_\theta\mathcal{A} & \xrightarrow{\ m^{1-1/\theta}\ } & S^0_1\mathcal{A} \end{array}$$

Where the embedding $S^0_\theta\mathcal{A} \to S^0_1\mathcal{A}$ is simply the Stechkin Lemma* [9].